On the ferromagnetism equations with large variations solutions

Olivier Guès, Franck Sueur

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Abstract

We exhibit some large variations solutions of the Landau-Lifschitz equations as the exchange coefficient ε^2 tends to zero. These solutions are described by some asymptotic expansions which involve some internals layers by means of some large amplitude fluctuations in a neighborhood of width $\sim \varepsilon$ of an hypersurface contained in the domain. Despite the nonlinear behaviour of these layers we manage to justify locally in time these asymptotic expansions.

1 Introduction

Ferromagnetic materials can attain a large magnetization under the action of a small applied magnetic field. To explain this phenomenon, in 1907, Weiss suggested that a *spontaneous magnetization* occurs. In 1928 Heisenberg explained the spontaneous magnetization postulated by Weiss in terms of the *exchange energy*. In 1935 Landau and Lifschitz (cf. [11]) proposed a quantitative theory, now known as *micromagnetics*. For a piece of ferromagnet-which is supposed to be a regular bounded open set Ω in $\mathbb{R}^3$ with a smooth boundary, and locally on one side of Γ-the magnetic state at a point $x \in \Omega$ at time t is given by a vector $u(t,x) \in \mathbb{R}^3$ which belongs to the unit sphere of $\mathbb{R}^3$, called the *magnetic moment*. The Landau-Lifschitz equations read:

$$\partial_t u^\varepsilon = u^\varepsilon \wedge (\mathcal{H}(u^\varepsilon) + \varepsilon^2 \Delta u^\varepsilon) - u^\varepsilon \wedge \big(u^\varepsilon \wedge (\mathcal{H}(u^\varepsilon) + \varepsilon^2 \Delta u^\varepsilon)\big) \quad \text{ in } \Omega, \tag{1.1}$$

where $\varepsilon > 0$ is the exchange coefficient. We denote $\mathcal{H}(u) := H_{|\Omega} \in L^2(\Omega;\mathbb{R}^3)$ where the magnetic field $H \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, is the unique solution of the following elliptic problem,

$$\begin{cases} H \in L^2(\mathbb{R}^3;\mathbb{R}^3) \ , \\ \text{curl } H = 0 \text{ in } \mathbb{R}^3 \ , \\ \text{div } (H + \overline{u}) = 0 \text{ in } \mathbb{R}^3 \ , \end{cases} \tag{1.2}$$

where $\overline{u}$ means the extension of u by 0 outside of the set Ω. The equations (1.1) are supplemented by the homogeneous Neumann boundary condition:

$$\partial_{\mathbf{n}} u^\varepsilon = 0 \ \text{ in } \ \Gamma, \tag{1.3}$$

where $\mathbf{n}$ is the unitary outward normal at the boundary Γ, and by an initial condition:

$$u^\varepsilon_{|t=0} = u_0. \tag{1.4}$$

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The solution must also satisfy the constraint

$$|u^\varepsilon(t,x)| = 1, \quad \text{ for } \ x \in \Omega, \ t \geq 0 \tag{1.5}$$

which is obviously propagated from the initial data as soon as it is satisfied at $t = 0$.

In this paper we study the asymptotic behaviour of the solutions of the Landau-Lifschitz equations (1.1)-(1.3)-(1.4) as the exchange coefficient ε tends to zero. From a formal point of view, when $\varepsilon = 0$, the system (1.1)-(1.3)-(1.4) becomes

$$\begin{cases} \partial_t u^0 = u^0 \wedge \mathcal{H}(u^0) - u^0 \wedge (u^0 \wedge \mathcal{H}(u^0)) \quad \text{ in } \Omega \\ \\ u^0_{|t=0} = u_0 \ , \end{cases} \tag{1.6}$$

where no boundary condition is needed. In the paper [4] it is proved that, for *smooth enough solutions* the system (1.6) is a "good approximation" of the full system (1.1)-(1.3)-(1.4) in the sense that the solution u^0 of (1.6) is indeed limit in $L^2([0,T] \times \Omega)$ of solutions u^ε of (1.1)-(1.3)-(1.4) as $\varepsilon \to 0$. However, this result holds under the assumption that u^0 belongs to the space $\mathcal{C}\big([0,T], H^5(\Omega)\big)$ where $H^5(\Omega)$ is the usual Sobolev space. In particular this assumption excludes the case where u^0 is *discontinuous* across an hypersurface contained in Ω and it was one motivation behind this paper to treat that case.

First of all, let us observe that the system (1.6) actually admits discontinuous solutions. To simplify, we will restrict the analysis to piecewise smooth solutions. We assume that Σ is a smooth compact hypersurface contained in Ω. For $0 \leq s < \infty$ call $H^s_\Sigma(\Omega)$ the set of functions $u \in L^2(\Omega)$ such that $u_{|\Omega_\pm} \in H^s(\Omega_\pm)$ where $H^s(\Omega_\pm)$ is the usual Sobolev space on L^2. We endow $H^s_\Sigma(\Omega)$ with the norm

$$\|u\|_{H^s_\Sigma} := \|u_{|\Omega_-}\|_{H^s(\Omega_-)} + \|u_{|\Omega_+}\|_{H^s(\Omega_+)}$$

This definition extends to the case when $s = \infty$: the space $H^\infty_\Sigma(\Omega)$ is the natural Fréchet space. We get the following result of global existence of solution of (1.6) discontinuous through the hypersurface Σ.

Theorem 1.1. *Let $s \in]\frac{3}{2}, \infty]$ and $u_0 \in H^s_\Sigma(\Omega)$. Then there exists a unique $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^s_\Sigma(\Omega)\big)$ solution of the Cauchy problem* (1.6).

Proof. This result can be easily obtained by following the proof of Proposition 4.1 of [4], with only a few adaptations. By the way in the closer setting of semilinear symmetric hyperbolic system, it is well known since the works of Métivier [12] that there exist some local piecewise regular solutions discontinuous across a smooth hypersurface which is a characteristic hypersurface of constant multiplicity for this hyperbolic system. In the present setting, the proof is in fact simpler since the hypersurface Σ is totally characteristic. Moreover thanks to (1.5) and since the operator $\mathcal{H}$ satisfies the transmission property, our setting allows to conclude to a global existence. □

Let us now claim a first theorem about the asymptotic behaviour of the solutions of the Landau-Lifschitz equations (1.1)-(1.3)-(1.4) as the exchange coefficient ε tends to zero.

Theorem 1.2. *Let $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^\infty_\Sigma(\Omega)\big)$ be a solution of* (1.6). *There exist $T > 0$ and a family of solutions $u^\varepsilon \in \mathcal{C}^\infty([0,T] \times \Omega)$, $\varepsilon \in]0,1]$, of the equation* (1.1) *on $[0,T] \times \Omega$, of the equation* (1.3) *on $[0,T] \times \Gamma$, such that there exist $C > 0$ and ε_0 such that for all $\varepsilon \in]0, \varepsilon_0]$,*

$$||u^\varepsilon - u^0||_{L^2([0,T]\times\Omega)} \leqslant C\varepsilon^{\frac{1}{2}}.$$

To begin with some comments about Theorem 1.2 let us stress that we do not prescribe the initial data (1.4) for the u^ε. Thus the traces of the u^ε at $t = 0$ are not equal in general to the trace of u^0 at $t = 0$. So Theorem 1.2 claims the existence of local in time solutions $u^\varepsilon \in \mathcal{C}^\infty$, of the equation (1.1) on Ω, of the equation (1.3) on Γ, which converge to u^0 as ε tends to zero in L^2, with a rate of convergence in $\varepsilon^{\frac{1}{2}}$.

Indeed, in this paper, we will claim a more accurate result in Theorem 2.1 by showing that the u^ε can be described with a WKB expansion which involves some boundary layers profiles. On one hand, a boundary layer appears near the boundary to compensate the lost of the Neumann condition from the complete model (1.1)-(1.3)-(1.4) to the limite model (1.6) ($\varepsilon = 0$). Such a boundary layer was already studied in paper [4]. The amplitude of this boundary is weak and its behaviour is linear. On another hand, there are some boundary layers on each side of the hypersurface Σ. Their task is to compensate the lost of transmission conditions across Σ from the complete model (1.1)-(1.3)-(1.4) to the limit model (1.6) ($\varepsilon = 0$).

Remark 1.1. Such an analysis is inspired by the paper [13] where we show that discontinuous solutions of multidimensional semilinear symmetric hyperbolic systems, which are regular outside of a smooth hypersurface characteristic of constant multiplicity, are limits, when $\varepsilon \to 0$, of solutions $(u^\varepsilon)_{\varepsilon\in]0,1]}$ of the system perturbated by a viscosity of size ε. In this paper, we adapt the method to the ferromagnetism quasi-static model, where in particular the non local operator $\mathcal{H}$ occurs. We point out that for the limit model ($\varepsilon = 0$), the hypersurface is totally characteristic. As a consequence, the analysis involves only characteristic boundary layers. On the opposite, [13] stresses the occurrence of characteristic and non characteristic boundary layers. It could be also possible -as in [13]- to study the case where the singularity is weaker than a jump of the function u^0 as a jump of a derivative of the function u^0. Then we can take T as big as we want and the quality of the approximation is as better as the jump concerns a higher order derivative. We also refer to papers [10], [9], [13] for the use of boundary layers in transmission strategy.

Remark 1.2. It could be interesting to know if it is possible to obtain such a result in the non static case for which the Landau-Lifschitz equation is coupled with the Maxwell system of electromagnetic. For such a model an analysis of the boundary layer induced by the Neumann boundary condition on Γ is performed in [5].

Remark 1.3. With the same method than the one used in this paper, it is possible to get global in time $O(\varepsilon^s)$ approximation for all $s < \frac{1}{2}$. More precisely for any $s < \frac{1}{2}$ there exists a family of solutions $u^\varepsilon \in \mathcal{C}^\infty([0, T_\varepsilon] \times \Omega)$, of the equation (1.1) on $[0, T_\varepsilon] \times \Omega$, of the equation (1.3) on $[0, T_\varepsilon] \times \Gamma$, $\varepsilon \in]0, 1]$, with $\lim_{\varepsilon\to 0^+} T_\varepsilon = \infty$, such that for all $T > 0$, there exists $C > 0$ and ε_0 such that for all $\varepsilon \in]0, \varepsilon_0]$, there holds $||u^\varepsilon - u^0||_{L^2([0,T]\times\Omega)} \leqslant C\varepsilon^s$.

2 Asymptotic expansion

Let us fix some notations. We will use the letter $\mathcal{S}$ to denote the Schwarz space of rapidly decreasing functions. We define the boundary layer profile spaces

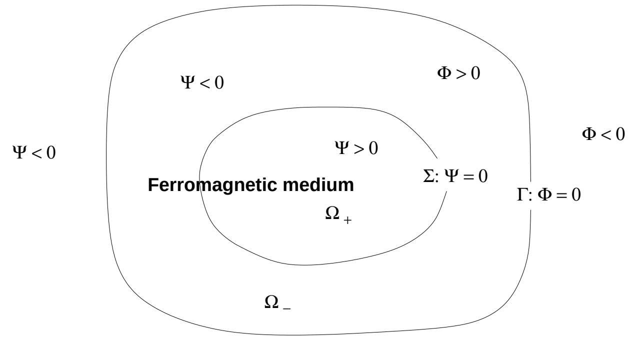
$$\mathcal{N}_\pm(T) := H^\infty([0, T] \times \Omega, \mathcal{S}(\mathbb{R}^\pm)).$$

Since we will need an equation of the boundary Γ, we fix once for all a function $\Phi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ and we assume that $\Omega = \{\Phi > 0\}$, $\Gamma = \{\Phi = 0\}$ and $|\nabla\Phi(x)| = 1$ in an open neighborhood $\mathcal{V}_\Gamma$ of Γ [2]. Let us also fix a function $\Psi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\Sigma = \{\Psi = 0\}$ and such that $|\nabla\Psi(x)| = 1$ in an open neighborhood $\mathcal{V}_\Sigma$ of Σ [3]. We assume that the neighborhoods $\mathcal{V}_\Gamma$ and

[2]Hence for $x \in \Omega \cap \mathcal{V}_\Gamma$: $\Phi(x) = dist(x, \Gamma)$.

[3]Hence for $x \in \Omega \cap \mathcal{V}_\Sigma$: $\psi(x) = dist(x, \Sigma)$.

$\mathcal{V}_\Sigma$ have been fixed small enough in order that $\mathcal{V}_\Gamma \cap \mathcal{V}_\Sigma = \varnothing$. We will denote $\Omega_+ := \Omega \cap \{\Psi > 0\}$ and $\Omega_- := \Omega \cap \{\Psi < 0\}$. We consider a C^∞ unit vector field $\partial_{\mathbf{n}}$ which coincides on $\mathcal{V}_\Gamma$ with $-\nabla_x \Phi \cdot \nabla_x$ and on $\mathcal{V}_\Sigma$ with $-\nabla_x \Psi \cdot \nabla_x$.



In the easier case where u^0 is continuous across the hypersurface Σ, paper [4] shows the existence of solutions u^ε, $\varepsilon \in]0,1]$, of the equation (1.1) in Ω, of the equation (1.3) on Γ, of the form

$$u^\varepsilon(t,x) \quad := \quad u^0(t,x) + \varepsilon\Big(\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon}) + \mathbf{w}^\varepsilon(t,x)\Big)$$

where the function $\mathfrak{U}$ is in $\mathcal{N}_+(\infty)$ and satisfies $\mathfrak{U}(t,x,z) = 0$ for $x \notin \mathcal{V}_\Gamma$. The function $\mathfrak{U}$ describes a boundary layer which appears near the boundary to compensate the lost of the Neumann condition from the complete model (1.1)-(1.3)-(1.4) to the limit model (1.6) ($\varepsilon = 0$). The amplitude of this boundary is weak and its behaviour is linear. For sake of completeness we will state this in section 2.2. The functions $\mathbf{w}^\varepsilon$ can be seen as remainders.

Here since we deal with a ground state u^0 which is discontinuous across the hypersurface Σ, we look for solutions u^ε, $\varepsilon \in]0,1]$, of the equation (1.1) in Ω, of the equation (1.3) on Γ, of the form

$$u^\varepsilon(t,x) := \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon}) + \varepsilon\Big(\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon}) + \mathbf{w}^\varepsilon(t,x)\Big). \tag{2.1}$$

The function $\mathcal{U}$ describes a large amplitude internal layer profile i.e. a sharp transition in the neighborhood of the hypersurface Σ of width $\sim \varepsilon$. More precisely the function $\mathcal{U}$ is $\mathcal{C}^\infty$ and satisfies

$$\lim_{y \to \pm\infty} \mathcal{U}(t,x,y) \quad = \quad u^0(t,x) \quad \text{for } x \in \mathcal{V}_\Sigma \cap \Omega_\pm \tag{2.2}$$

$$\mathcal{U}(t,x,y) \quad = \quad u^0(t,x) \quad \text{for } x \notin \mathcal{V}_\Sigma \text{ and } y \in \mathbb{R} \tag{2.3}$$

The profile $\mathfrak{U}$, as we have already said it above, was constructed in [4]. The functions $\mathbf{w}^\varepsilon$ can still be seen as remainders. Let us explain this time more precisely what we mean by remainders. Let us fix a finite set of smooth vectors fields $\mathcal{T}_0 = \{\mathcal{Z}_i(x;\partial_x); i = 1,\cdots,\mu\}$ on $\mathbb{R}^3$, tangent to the surfaces Γ and Σ (that is satisfying $\mathcal{Z}_i(x;\partial_x)\Phi = 0$ on Γ and $\mathcal{Z}_i(x;\partial_x)\Psi = 0$ on Σ, for all $i \in \{1,\cdots,\mu\}$), and generating the algebra of smooth vector fields tangent to $\Gamma \cup \Sigma$. These vector fields can be viewed as vector fields on $\mathbb{R}^4$ tangent to $\mathbb{R} \times \Gamma$ and to $\mathbb{R} \times \Sigma$. By adding the vector field ∂_t to the family, one gets the set $\mathcal{T} := \{\partial_t\} \cup \mathcal{T}_0$ which generates the set of smooth vector fields in $\mathbb{R}^4$ tangent to $(\mathbb{R}\times\Gamma) \cup (\mathbb{R}\times\Sigma)$. We denote $\mathcal{Z}_0 := \partial_t$. For all multi-index $\alpha \in \mathbb{N}^{1+\mu}$ we note $\mathcal{Z}^\alpha = \partial_t^{\alpha_0}\mathcal{Z}_1^{\alpha_1}.\cdots.\mathcal{Z}_\mu^{\alpha_\mu}$, with $\alpha = (\alpha_0,\alpha_1,\cdots,\alpha_\mu)$. Let us introduce the usual norm:

$$\|u\|_m := \sum_{|\alpha| \le m,\, \alpha \in \mathbb{N}^{1+\mu}} |\mathcal{Z}^\alpha u\|_{L^2(]0,T[\times\Omega)},$$

and note $H^m_{co}(]0,T[\times\Omega)$ the space of $u \in L^2(]0,T[\times\Omega)$ such that this norm is finite. We introduce the set E of the family $(\mathbf{w}^\varepsilon)_{0<\varepsilon\leqslant 1})$ of functions in $L^2(]0,T[\times\Omega)$ such that for all $m \in \mathbb{N}$, there exists $\varepsilon_0 > 0$ such that

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} (||\mathbf{w}^\varepsilon||_m + ||\varepsilon\partial_{\mathbf{n}}\,\mathbf{w}^\varepsilon||_m + \varepsilon(||\mathbf{w}^\varepsilon||_\infty + ||\mathcal{Z}\mathbf{w}^\varepsilon||_\infty + ||\varepsilon\partial_{\mathbf{n}}\,\mathbf{w}^\varepsilon||_\infty)) < \infty. \tag{2.4}$$

In fact Theorem 1.2 is the straightforward consequence of the following result.

Theorem 2.1. *Let $u^0 \in \mathcal{C}^\infty\big(\mathbb{R}, H^\infty_\Sigma(\Omega)\big)$ be a solution of* (1.6)*. There exist $T > 0$, a profile $\mathcal{U}$ in $\mathcal{C}^\infty((0,T)\times\Omega\times\mathbb{R})$ which satisfies* (2.2) − (2.3) *and a family* $(\mathbf{w}^\varepsilon)$ *in E such that the function u^ε given by the formula (2.1) are solutions in C^∞ of the equation* (1.1) *on* $[0,T]\times\Omega$*, of the equation* (1.3) *on* $[0,T]\times\Gamma$*.*

Theorem 2.1 exhibits some large variations solutions of the Landau-Lifschitz equations as the exchange coefficient ε^2 tends to zero, by means of the asymptotic expansions (2.1). The remainder of the paper is devoted to the proof of Theorem 2.1. As in [4], since the magnetic moment u is unimodular, the equation (1.1) is equivalent for smooth solutions to the following one:

$$\mathcal{L}^\varepsilon(u^\varepsilon,\partial)\,u^\varepsilon = \mathbf{F}\big(u^\varepsilon, \varepsilon\partial_x u^\varepsilon, \mathcal{H}(u^\varepsilon)\big) \tag{2.5}$$

where we have noted

$$\mathcal{L}^\varepsilon(v,\partial) := \partial_t - \varepsilon^2\Delta_x - \varepsilon^2 v \wedge \Delta_x,$$

and

$$\mathbf{F}(u,V,H) := |V|^2 u + u\wedge H - u\wedge(u\wedge H),$$

for all $u \in \mathbb{R}^3$, $V \in \mathcal{M}(\mathbb{R}^3,\mathbb{R}^3)$, $H \in \mathbb{R}^3$. From now on we will deal with equation (2.5) rather than (1.1). We will proceed in three steps. In subsection 2.1 we will define the profile $\mathcal{U}$ as a local in time solution of a pair of nonlinear equations in $\Omega\times\mathbb{R}_\pm$ coupled by some transmissions conditions on $\{y = 0\}$. In subsection 2.2 we will recall the results of [4] about the profile $\mathfrak{U}$. In subsection 2.3 we will prove the existence of some remainders $\mathbf{w}^\varepsilon$ till the lifetime T of the profile $\mathcal{U}$. Eventually we will show that the remainders $\mathbf{w}^\varepsilon$ satisfy the uniform estimates uniform (2.4).

2.1 Construction of the internal layers

Despite that $\pm\frac{\Psi(x)}{\varepsilon} > 0$ when $x \in \Omega_\pm$ we will define $\mathcal{U}$ for all $(x,z) \in \Omega\times\mathbb{R}_\pm$ since this will not cause any additional difficulty. An Uryshon argument yields the existence of two functions $u^0_\pm$ in $H^\infty((0,\infty)\times\Omega)$ such that $u^0_\pm = u^0$ for all $x \in \Omega_\pm \cup (\Omega_\mp - \mathcal{V}_\Sigma)$. We look for a viscous internal layer profile $\mathcal{U}$ of the form

$$\mathcal{U}(t,x,y) := \begin{cases} u^0_+(t,x) + \mathcal{U}_+(t,x,y) & \text{if } y > 0, \\ u^0_-(t,x) + \mathcal{U}_-(t,x,y) & \text{if } y < 0. \end{cases} \tag{2.6}$$

The functions $\mathcal{U}_\pm$ are in $\mathcal{N}_\pm(T)$. These functions describe some internal large amplitude boundary layers, on each side of the hypersurface Σ. To insure that the function $\mathcal{U}$ is in $C^1((0,T)\times\Omega\times\mathbb{R})$ it is necessary to impose the transmission conditions:

$$\left.\begin{array}{c} \mathcal{U}_+ - \mathcal{U}_- = -u^0_+ + u^0_-, \\ \partial_y\mathcal{U}_+ - \partial_y\mathcal{U}_- = 0 \end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}. \tag{2.7}$$

In Theorem 2.2 we will define the profiles $\mathcal{U}_\pm$ as local solutions of nonlinear equations in $\Omega \times \mathbb{R}_\pm$ coupled by some transmissions conditions on $\{y = 0\}$. Let us look for convenient equations. We will plug the functions $u^{\varepsilon,0}$ defined by $u^{\varepsilon,0}(t,x) := \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon})$ instead of u^ε in (2.5). In general it is not possible to verify (2.5) but we will try to choose the functions $\mathcal{U}_\pm$ such that the error term is as small as possible. Let us begin to look at the left side of (2.5). With (2.6) we get in L^∞

$$\mathcal{L}^\varepsilon(u^{\varepsilon,0},\partial)u^{\varepsilon,0} = \partial_t\, u^0_\pm + \Big(L(\mathcal{U},\partial_t,\partial_y^2)\mathcal{U}_\pm\Big)| + O(\varepsilon) \quad \text{for } x \in \Omega_\pm, \tag{2.8}$$

where the vertical bar $|$ means that y is evaluated in $y = \frac{\Psi(x)}{\varepsilon}$ and

$$L(U,\partial_t,\partial_y^2) := \partial_t - \partial_y^2 - U \wedge \partial_y^2.$$

We now turn to the right side of (2.5). We first look at the action of $\mathcal{H}$ on the family $u^{\varepsilon,0}$:

$$\mathcal{H}(u^{\varepsilon,0}) = \mathcal{H}(u^0_\pm) - (\mathcal{U}_\pm.n)|\, n + O(\varepsilon).$$

Then

$$\mathbf{F}\big(u^\varepsilon, \varepsilon\partial_x u^\varepsilon, \mathcal{H}(u^\varepsilon)\big) := \mathbf{F}\big(u^0_\pm, 0, \mathcal{H}(u^0_\pm)\big) + F_\pm(\mathcal{U}_\pm, \partial_y\mathcal{U}_\pm)| + O(\varepsilon) \quad \text{for } x \in \Omega_\pm, \tag{2.9}$$

with for all $U \in \mathbb{R}^3$, $V \in \mathcal{M}(\mathbb{R}^3,\mathbb{R}^3)$,

$$\begin{aligned} F_\pm(U,V) \ &:= \ |V|^2\,(u^0_\pm + U) + U \wedge \mathcal{H}(u^0_\pm) - (U.n)(u^0_\pm + U)\wedge n \\ &\quad + U \wedge (u^0_\pm + U) \wedge (\mathcal{H}(u^0_\pm) - (U.n)n) + u^0_\pm \wedge (U \wedge (\mathcal{H}(u^0_\pm) - (U.n)n)) \\ &\quad - (U.n)u^0_\pm \wedge (u^0_\pm \wedge n) \end{aligned}$$

Thanks to (2.8) and (2.9) we get by looking at the terms at order 0

$$\partial_t\, u^0_\pm + L(\mathcal{U},\partial_t,\partial_y^2)\mathcal{U}_\pm = \mathbf{F}\big(u^0_\pm, 0, \mathcal{H}(u^0_\pm)\big) + F_\pm(\mathcal{U}_\pm, \partial_y\mathcal{U}_\pm).$$

Since for $x \in \Omega_\pm$, the functions $u^0_\pm$ satisfies (1.6) we could simplify and we get the nonlinear equations

$$L(u^0_\pm + \mathcal{U}_\pm, \partial_t, \partial_y^2)\mathcal{U}_\pm = F_\pm(\mathcal{U}_\pm, \partial_y\mathcal{U}_\pm). \tag{2.10}$$

The equations (2.10) are parabolic with respect to t, y, the variable x can be seen as a parameter. The following theorem claims that it is possible to find some solutions $\mathcal{U}_\pm \in \mathcal{N}_\pm(T)$ of these equations even for all $x \in \Omega$.

Theorem 2.2. *There exists $T > 0$ and there exist some functions $\mathcal{U}_\pm \in \mathcal{N}_\pm(T)$ which verify the equations* (2.10) *when $(t,x,y) \in (0,T)\times\Omega\times\mathbb{R}_\pm$ and the transmission conditions* (2.7). *Moreover precisely for all $x \notin \mathcal{V}_\Sigma$ and $y \in \mathbb{R}_\pm$ there holds $\mathcal{U}_\pm(t,x,y) = 0$.*

Proof. We will proceed in four steps.

Step 1. *We begin to reduce the problem to homogeneous boundary conditions.*

We introduce the functions $V_\pm$ and $\mathcal{U}_\pm$ given by the formula

$$\begin{aligned} V_\pm(t,x,y) \ &:= \ (1 - \frac{e^{\mp y}}{2})u^0_\pm(t,x) + \frac{e^{\mp y}}{2} u^0_\mp(t,x), \\ \mathbf{W}_\pm(t,x,y) \ &:= \ \mathcal{U}_\pm(t,x,y) \pm \frac{1}{2}(u^0_+(t,x) - u^0_-(t,x))e^{\mp y}. \end{aligned}$$

Thus the transmission conditions (2.7) reads:

$$\left.\begin{array}{r}\mathbf{W}_+ - \mathbf{W}_- = 0,\\ \partial_y \mathbf{W}_+ - \partial_y \mathbf{W}_- = 0\end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}. \tag{2.11}$$

Moreover the equations (2.10)-(2.7) read for $(t,x,y) \in (0,T)\times\Omega\times\mathbb{R}_\pm$:

$$L(V_\pm + \mathbf{W}_\pm, \partial_t, \partial_y^2)\mathbf{W}_\pm \quad = \quad \hat{F}_\pm(t,x,y,\mathbf{W}_\pm,\partial_y\mathbf{W}_\pm), \tag{2.12}$$

where $\hat{F}_\pm$ are C^∞ functions such that the functions $\hat{F}_\pm(t,x,y,0,0)$ are rapidly decreasing with respect to y.

Step 2. *We prove the existence of compatible initial data.*

Let us to explain why the initial values $\mathbf{W}_{0,+}$ must satisfy some compatibility conditions at the corner $\{t=y=0\}$ are required in order to obtain smooth solutions $\mathbf{W}_\pm$ of the problem (2.12)-(2.11) with $\mathbf{W}_\pm|_{t=0} := \mathbf{W}_{0,\pm}$. We start with the condition of order 0. Set $t=0$ in the transmission conditions (2.11) to see that $\mathbf{W}_{0,+}$ must satisfy the relation

$$\left.\begin{array}{r}\mathbf{W}_+ - \mathbf{W}_{0,-} = 0,\\ \partial_y \mathbf{W}_{0,+} - \partial_y \mathbf{W}_{0,-} = 0\end{array}\right\} \quad \text{when } (x,y) \in \Omega\times\{0\}. \tag{2.13}$$

Now, for each $k \geqslant 1$, apply the derivative ∂_t^k to the transmission conditions (2.7). We get

$$\left.\begin{array}{r}\partial_t^k\mathbf{W}_+ - \partial_t^k\mathbf{W}_- = 0,\\ \partial_y\partial_t^k \mathbf{W}_+ - \partial_y\partial_t^k \mathbf{W}_- = 0\end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}.$$

Now remark that, by iteration, we can extirpate $\partial_t^k\mathbf{W}_\pm$ by the interior equations (2.10) in terms of derivatives with respect to y. More precisely there exists some smooth functions $C_\pm^k$ such that $\partial_t^k\mathbf{W}_\pm = C_\pm^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k})$. Thus the following kth order compatibility condition must hold:

$$\left.\begin{array}{r}C_+^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k}) - C_-^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k}) = 0,\\ \partial_y C_+^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k}) - \partial_y C_-^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k}) = 0.\end{array}\right\} \quad \text{when } (x,y) \in \Omega\times\{0\}. \tag{2.14}$$

Lemma 2.1. *There exist some initial values $\mathbf{W}_{0,\pm}$ in $H^\infty(\Omega,\mathcal{S}(\mathbb{R}_\pm))$ which satisfy the relation (2.13) and (2.14) for all $k \geqslant 1$.*

Proof. As we will follows the method of [13], we only sketch the proof for sake of completeness. We start by analyzing more accurately the compatibility conditions and more especially the way the functions $C_\pm^k$ depend on the derivatives with respect to y. Indeed they exists some functions $\tilde{C}_\pm^k$ such that

$$C_\pm^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k}) = \tilde{C}_\pm^k((\partial_y^l\mathbf{W}_\pm)_{l\leqslant 2k-1}) + (\partial_y^{2k} + (V_\pm + \mathbf{W}_\pm)\wedge\partial_y^{2k})\mathbf{W}_\pm.$$

Since given two functions $\mathbf{W}_\pm^{(0)}$ in $H^\infty(\Omega)$ the applications

$$\mathbf{W}_\pm \mapsto \mathbf{W}_\pm + (V_\pm + \mathbf{W}_\pm^{(0)})\wedge\mathbf{W}_\pm$$

are two automorphisms of $H^\infty(\Omega)$ and an iteration, we deduce by iteration that there exists a family $(\mathbf{W}_\pm^{(k)})_{k\in\mathbb{N}}$ in $H^\infty(\Omega)$ such that

$$\begin{array}{c}C_+^k((\partial_y\mathbf{W}_\pm^{(l)})_{l\leqslant 2k}) - C_-^k((\partial_y\mathbf{W}_\pm^{(l)})_{l\leqslant 2k}) = 0,\\ \partial_y C_+^k((\partial_y\mathbf{W}_\pm^{(l)})_{l\leqslant 2k}) - \partial_y C_-^k((\partial_y\mathbf{W}_\pm^{(l)})_{l\leqslant 2k}) = 0.\end{array}$$

We end the proof by a classical Borel argument. $\square$

As a consequence, we will assume in the rest of the proof that the functions $\mathbf{W}_\pm$ vanish for $t \leqslant 0$.

Step 3. *We look for linear estimates.*

In order to use an iterative scheme, we look at the linear problem

$$L(\mathfrak{W}_\pm, \partial_t, \partial_y^2)\mathbf{W}_\pm = f_\pm \quad \text{when } (t, x, y) \in (0, T) \times \Omega \times \mathbb{R}_\pm, \tag{2.15}$$

$$\left.\begin{array}{r}\mathbf{W}_+ - \mathbf{W}_- = 0, \\ \partial_y \mathbf{W}_+ - \partial_y \mathbf{W}_- = 0\end{array}\right\} \quad \text{when } (t, x, y) \in (0, T) \times \Omega \times \{0\}. \tag{2.16}$$

For all real $\lambda \geqslant 1$, the space $L^2((0, T) \times \Omega \times \mathbb{R}_\pm)$ is endowed with the scalar product associated to the Euclidean norm

$$||\mathbf{W}_\pm||_{0,\lambda,T} := ||e^{-\lambda t}\, W_\pm||_{L^2((0,T)\times\Omega\times\mathbb{R}_\pm)}$$

In order to avoid heavy notations, we will denote $W := (\mathbf{W}_+, \mathbf{W}_-)$, $f := (f_+, f_-)$ and $\mathfrak{W} := (\mathfrak{W}_+, \mathfrak{W}_-)$. We endow the space $L^2((0, T) \times \Omega \times \mathbb{R}_+) \times L^2((0, T) \times \Omega \times \mathbb{R}_-)$ with the scalar product associated to the Euclidean norm

$$||W||_{0,\lambda,T} := ||\mathbf{W}_+||_{+,0,\lambda,T} + ||\mathbf{W}_-||_{-,0,\lambda,T}.$$

For $m \in \mathbb{N}$, we introduce the following weighted norms:

$$||W||_{m,\lambda,T} := \sum_{|\alpha| \leqslant m} ||\partial_{t,x}^\alpha\, W||_{0,\lambda,T}, \text{ and } |W|_{m,\lambda,T} := \sum_{|\alpha| \leqslant m} ||\partial_{t,x}^\alpha\, \partial_y^{\alpha_4}\, W||_{0,\lambda,T},$$

where $\alpha := (\alpha_0, ..., \alpha_3) \in \mathbb{N}^4$ and $\partial_{t,x}^\alpha := \partial_t^{\alpha_0} \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3}$.

Proposition 2.1. *Let $R > 0$. If $\mathfrak{W}_\pm$ verify the following estimates*

$$||\mathfrak{W}_+||_{Lip((0,T)\times\Omega\times\mathbb{R}_+)} + ||\mathfrak{W}_+||_{Lip((0,T)\times\Omega\times\mathbb{R}_+)} + |\mathfrak{W}|_{m,\lambda,T} < R,$$

and the following boundary conditions

$$\left.\begin{array}{r}\mathfrak{W}_+ - \mathfrak{W}_- = 0, \\ \partial_y \mathfrak{W}_+ - \partial_y \mathfrak{W}_- = 0\end{array}\right\} \quad \text{when } (t, x, y) \in (0, T) \times \Omega \times \{0\}, \tag{2.17}$$

then there exist $\lambda_m > 0$ and for all $k \in \mathbb{N}$, $\mu_{k,m} > 0$, such that for all $\lambda \geqslant \lambda_m$,

$$|W|_{m,\lambda,T} \leqslant \frac{\lambda_m}{\lambda} |f|_{m,\lambda,T} \tag{2.18}$$

and for all $\mu \geqslant \mu_{k,m}$,

$$|y^k\, W|_{m,\lambda,T} \leqslant \frac{\mu_{k,m}}{\mu} \sum_{j=0}^{k} |y^j\, f|_{m,\mu,T}. \tag{2.19}$$

Proof. We multiply the equation (2.15) by $\mathbf{W}_\pm$ and integrate for $(x, y) \in \Omega \times \mathbb{R}_\pm$. Hence

$$(1/2)\partial_t \int_{\Omega\times\mathbb{R}_\pm} |\mathbf{W}_\pm|^2 - J_{1,\pm} - J_{2,\pm} = \int_{\Omega\times\mathbb{R}_\pm} f_\pm.\mathbf{W}_\pm \tag{2.20}$$

$$\text{where } J_{1,\pm} := \int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.\partial_y^2 \mathbf{W}_\pm \text{ and } J_{2,\pm} := \int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.(\mathfrak{W}_\pm \wedge \partial_y^2)\mathbf{W}_\pm.$$

Integrating by parts, we get

$$J_{1,\pm} = -\int_{\Omega\times\mathbb{R}_\pm} |\partial_y \mathbf{W}_\pm|^2 - I_{1,\pm}, \text{ and } J_{2,\pm} = -\int_{\Omega\times\mathbb{R}_\pm} \mathbf{W}_\pm.(\partial_y \mathfrak{W}_\pm \wedge \partial_y)\mathbf{W}_\pm - I_{2,\pm},$$
$$\text{where } I_{1,\pm} := \int_\Omega (\mathbf{W}_\pm.\partial_y \mathbf{W}_\pm)|_{y=0}, \text{ and } I_{2,\pm} := \int_\Omega (\mathbf{W}_\pm.(\mathfrak{W}_\pm \wedge \partial_y \mathbf{W}_\pm))|_{y=0}.$$

Using the boundary conditions (2.16) and (2.17), we get $I_{1,+} - I_{1,-} = I_{2,+} - I_{2,-} = 0$. Taking that into account we add the two estimates in (2.20). Then we multiply by $e^{-2\lambda t}$ and integrate in time. By a Gronwall lemma we get that there exists $c > 0$ such that for all $\lambda \geqslant c$,

$$|\partial_y W|^2_{0,\lambda,T} + \lambda |W|^2_{0,\lambda,T} \leqslant c| < f, W >_{\lambda,T} |. \tag{2.21}$$

We go on with estimates tangential to $\{y = 0\}$. To do this we apply the derivative $\partial^\alpha_{t,x}$ to the equations (2.15)-(2.16). So we get that $\partial^\alpha_{t,x} \mathbf{W}_\pm$ verify

$$L(\mathfrak{W}_\pm, \partial_t, \partial_y^2)\partial^\alpha_{t,x} \mathbf{W}_\pm = \tilde{f}_\pm \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\mathbb{R}_\pm, \tag{2.22}$$

$$\left.\begin{array}{r} \partial^\alpha_{t,x} \mathbf{W}_+ - \partial^\alpha_{t,x} \mathbf{W}_- = 0, \\ \partial_y\partial^\alpha_{t,x} \mathbf{W}_+ - \partial_y\partial^\alpha_{t,x} \mathbf{W}_- = 0 \end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}, \tag{2.23}$$

where

$$\tilde{f}_\pm := \partial^\alpha_{t,x} f_\pm + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|, |\alpha_2|<|\alpha|} \partial^{\alpha_1}_{t,x} \mathfrak{W}_\pm \wedge \partial_y^2 \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm. \tag{2.24}$$

We apply the tangential derivative $\partial^\alpha_{t,x}$ to the boundary conditions (2.17) and get

$$\left.\begin{array}{r} \partial^\alpha_{t,x} \mathfrak{W}_+ - \partial^\alpha_{t,x} \mathfrak{W}_- = 0, \\ \partial_y\partial^\alpha_{t,x} \mathfrak{W}_+ - \partial_y\partial^\alpha_{t,x} \mathfrak{W}_- = 0 \end{array}\right\} \quad \text{when } (t,x,y) \in (0,T)\times\Omega\times\{0\}, \tag{2.25}$$

Using the estimate (2.21), we get, for all $\lambda \geqslant c$,

$$|\partial_y\partial^\alpha_{t,x} W|^2_{0,\lambda,T} + \lambda |\partial^\alpha_{t,x} W|^2_{0,\lambda,T} \leqslant c| < \tilde{f}, \partial^\alpha_{t,x} W >_{\lambda,T} |.$$

Thanks to (2.24), we get

$$< \tilde{f}, \partial^\alpha_{t,x} W >_{\lambda,T} = < \partial^\alpha_{t,x} f, \partial^\alpha_{t,x} W >_{\lambda,T} + \sum_{|\alpha_1|+|\alpha_2|=|\alpha|, |\alpha_2|<|\alpha|} I_{\alpha_1,\alpha_2}, \tag{2.26}$$

where $I_{\alpha_1,\alpha_2} := I_{+,\alpha_1,\alpha_2} + I_{-,\alpha_1,\alpha_2}$ with

$$I_{\pm,\alpha_1,\alpha_2} := < \partial^{\alpha_1}_{t,x} \mathfrak{W}_\pm \wedge \partial_y^2 \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm, \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm >_{\lambda,T}.$$

Using Cauchy-Schwarz inequality, we get

$$| < \partial^\alpha_{t,x} f, \partial^\alpha_{t,x} W >_{\lambda,T} | \leqslant |f|_{0,\lambda,T}.|W|_{0,\lambda,T}.$$

We are going to estimate, for all α_1, α_2 such that $|\alpha_1| + |\alpha_2| = |\alpha|, |\alpha_2| < |\alpha|$, the term I_{α_1,α_2}. Integrating by parts, we get $I_{\pm,\alpha_1,\alpha_2} := \sum_{l=1}^3 I^l_{\pm,\alpha_1,\alpha_2}$, with

$$\begin{aligned}
I^1_{\pm,\alpha_1,\alpha_2} &:= - < \partial^{\alpha_1}_{t,x} \partial_y \mathfrak{W}_\pm \wedge \partial_y \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm, \partial^\alpha_{t,x} \mathbf{W}_\pm >_{\lambda,T}, \\
I^2_{\pm,\alpha_1,\alpha_2} &:= - < \partial^{\alpha_1}_{t,x} \mathfrak{W}_\pm \wedge \partial_y \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm, \partial^\alpha_{t,x} \partial_y \mathbf{W}_\pm >_{\lambda,T}, \\
I^3_{\pm,\alpha_1,\alpha_2} &:= \mp << \{(\partial^{\alpha_1}_{t,x} \mathfrak{W}_\pm \wedge \partial_y \partial^{\alpha_2}_{t,x} \mathbf{W}_\pm)\}|_{y=0}, \{\partial^\alpha_{t,x} \partial_y \mathbf{W}_\pm\}|_{y=0} >>_{\lambda,T},
\end{aligned}$$

where $<< .,. >>_{\lambda,T}$ denotes the scalar product of $L^2((0,T)\times\Omega)$ associated to the mesure $e^{-\lambda t}dtdx$. Thanks to the boundary conditions (2.23)-(2.25), we get $I^3_{+,\alpha_1,\alpha_2} - I^3_{-,\alpha_1,\alpha_2} = 0$. Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|I^1_{\pm,\alpha_1,\alpha_2}| &\leqslant |\partial^{\alpha_1}_{t,x}\,\partial_y \mathfrak{W}_\pm \wedge \partial_y\,\partial^{\alpha_2}_{t,x}\,\mathbf{W}_\pm|_{0,\lambda,T}\,.||\mathbf{W}_\pm||_{m,\lambda,T},\\
|I^2_{\pm,\alpha_1,\alpha_2}| &\leqslant |\partial^{\alpha_1}_{t,x}\,\mathfrak{W}_\pm \wedge \partial_y\,\partial^{\alpha_2}_{t,x}\,\mathbf{W}_\pm|_{0,\lambda,T}\,.||\partial_y\mathbf{W}_\pm||_{m,\lambda,T},
\end{aligned}
$$

Using Gargliardo-Nirenberg inequalities, we get

$$
\begin{aligned}
|I^1_{\pm,\alpha_1,\alpha_2}| &\leqslant c(||\partial_y\mathfrak{W}_\pm||_{m,\lambda,T}.||\mathbf{W}_\pm||_{Lip} + ||\mathfrak{W}_\pm||_{Lip}.||\partial_y\mathbf{W}_\pm||_{m,\lambda,T}).||\mathbf{W}_\pm||_{m,\lambda,T},\\
|I^2_{\pm,\alpha_1,\alpha_2}| &\leqslant c(||\mathfrak{W}_\pm||_{m-1,\lambda,T}.||\mathbf{W}_\pm||_{Lip} + |\mathfrak{W}_\pm|_{Lip}.||\partial_y\mathbf{W}_\pm||_{m-1,\lambda,T}).||\partial_y\mathbf{W}_\pm||_{m,\lambda,T}.
\end{aligned}
$$

Hence we get

$$
|I_{\alpha_1,\alpha_2}| \leqslant \frac{1}{2}||\partial_y\mathbf{W}_\pm||^2_{m,\lambda,T} + C(||\mathbf{W}_\pm||^2_{m,\lambda,T} + ||\partial_y\mathbf{W}_\pm||^2_{m-1,\lambda,T}).
$$

We deduce that there exists $\lambda_m > 0$ such that for all $\lambda \geqslant \lambda_m$, there holds $||W||_{m,\lambda,T} \leqslant \frac{\lambda_m}{\lambda}||f||_{m,\lambda,T}$.

To prove the estimates (2.18), it remains to get normal estimates. The cases $\alpha_4 = 0$ or 1 are already treated in the tangential estimates. If $\alpha_4 \geqslant 2$, we proceed by iteration, extirpating $\partial_y^2\mathbf{W}_\pm$ from the equations.

It remains to get the estimates (2.19). First we notice that for $p \geqslant 1$ the function $y^p\,\mathbf{W}_\pm$ verify the initial boundary value problem

$$
\begin{aligned}
&L(\mathfrak{W}_\pm,\partial_t,\partial_y^2)\mathbf{W}^{[p]}_\pm = f^{[p]}_\pm \quad \text{when } (t,x,y)\in(0,T)\times\Omega\times\mathbb{R}_\pm,\\
&\left.\begin{aligned}\mathbf{W}^{[p]}_+ - \mathbf{W}^{[p]}_- = 0,\\ \partial_y\mathbf{W}^{[p]}_+ - \partial_y\mathbf{W}^{[p]}_- = 0\end{aligned}\right\} \quad \text{when } (t,x,y)\in(0,T)\times\Omega\times\{0\},\\
&\mathbf{W}^{[p]}_\pm = 0 \quad \text{when } (t,x,y)\in\{0\}\times\Omega\times\mathbb{R}_\pm,
\end{aligned}
$$

where

$$
f^{[p]}_\pm \;=\; y^p\,f_\pm + \sum_{j=0}^{p-1}(q^1_j\partial_y\mathbf{W}^{[j]}_\pm + q^2_j\,\mathfrak{W}_\pm\wedge\partial_y\mathbf{W}^{[j]}_\pm),
$$

where the q^1_j and the q^2_j are in $\mathbb{N}$. Thus we prove, by iteration on p and thanks to the inequality (2.18), the estimate

$$
\sqrt{\mu}||\partial_y(y^p\,W)||_{m,\mu,T} + \mu||y^p\,W||_{m,\mu,T} \leqslant \sum_{j=0}^{p}||y^j\,f||_{m,\mu,T}
$$

which implies the estimate (2.19).

□

Step 4. *We use an iterative scheme.*

We define the iterative scheme $(\mathbf{W}^\nu_\pm)_{\nu\in\mathbb{N}}$ by setting $\mathbf{W}^0_\pm$ equal to zero and, by iteration, when $\mathbf{W}^\nu_\pm$ is defined, we take $\mathbf{W}^{\nu+1}_\pm$ as solution of

$$
\begin{aligned}
&L(V_\pm+\mathbf{W}^\nu_\pm,\partial_t,\partial_y^2)\mathbf{W}^{\nu+1}_\pm = \hat{F}(t,x,y,\mathbf{W}^\nu_\pm,\partial_y\mathbf{W}^\nu_\pm) \quad \text{when } (t,x,y)\in(0,\infty)\times\Omega\times\mathbb{R}_\pm,\\
&\left.\begin{aligned}\mathbf{W}^{\nu+1}_+ - \mathbf{W}^{\nu+1}_- = 0,\\ \partial_y\mathbf{W}^{\nu+1}_+ - \partial_y\mathbf{W}^{\nu+1}_- = 0\end{aligned}\right\} \quad \text{when } (t,x,y)\in(0,T)\times\Omega\times\{0\},\\
&\mathbf{W}^{\nu+1}_\pm = 0 \quad \text{when } (t,x,y)\in\{0\}\times\Omega\times\mathbb{R}_\pm.
\end{aligned}
$$

Thanks to the linear estimates, to a Sobolev embedding and to some Gargliardo-Nirenberg inequalities, we show that the iterative scheme $(\mathbf{W}_{\pm}^{\nu})_{\nu\in\mathbb{N}}$ converge, when $\nu \to +\infty$ toward some solutions $\mathbf{W}_{\pm} \in \mathcal{N}_{\pm}(T)$ of the problem (2.12)-(2.11). By going back to the original problem (2.10)-(2.7), the first sentence of Theorem 2.2 is now proved. When $x \notin \mathcal{V}_{\Sigma}$, the function $u_{+}^{0} - u_{-}^{0}$ in the right hand side of (2.10) vanishes and so do the functions $\mathcal{U}_{\pm}$.

□

Remark 2.1. Notice that the possibility of a blow-up can be controlled with Lipschitz norm in a very classical way. However we do not know whether the solutions $\mathcal{U}$ actually blow-up or globally exist.

2.2 Construction of $\mathfrak{U}$

In this section we define the boundary layer profile $\mathfrak{U}$ as a solution of a linear boundary value problem. Let us recall that this function describes a boundary layer which appears near the boundary to compensate the lost of the Neumann condition from the complete model (1.1)-(1.3)-(1.4) to the limit model (1.6) ($\varepsilon = 0$). Such a boundary layer was already mentioned in paper [4]. Let Θ be a C^{∞} function on Ω such that $\Theta = 1$ in a neighborhood $\mathcal{W}_{\Gamma}$ of Γ such that $\mathcal{W}_{\Gamma} \subset\subset \mathcal{V}_{\Gamma}$ and $\Theta = 0$ in $\Omega - \mathcal{W}_{\Gamma}$.

Theorem 2.3. *There exists $\mathfrak{U} \in \mathcal{N}_{+}(T)$ which verifies*

$$\begin{aligned} L(u^0, \partial_t, \partial_z^2)\mathfrak{U} &= -(\mathfrak{U}.n)u^0 \wedge n + \mathfrak{U} \wedge \mathcal{H}(u^0) \\ &\quad + \mathfrak{U} \wedge (u^0 \wedge \mathcal{H}(u^0)) - (\mathfrak{U}.n)u^0 \wedge (u^0 \wedge n) + u^0 \wedge (\mathfrak{U} \wedge \mathcal{H}(u^0), \end{aligned}$$

when $(t, x, z) \in (0, T) \times \Omega \times \mathbb{R}_{+}$,

$$\partial_z \mathfrak{U} = \Theta(x)\partial_{\mathrm{n}} u^0 \quad \text{when } (t, x, z) \in (0, T) \times \Omega \times \{0\}. \tag{2.27}$$

Moreover there holds $\mathfrak{U}(t, x, z) = 0$ for $x \notin \mathcal{V}_{\Sigma}$.

Proof. Proceeding as in the proof of Theorem 2.2, we prove the existence of compatible initial data. Then we follow the proof of Proposition 4.2 of [4]. □

2.3 Construction of $\mathbf{w}^{\varepsilon}$

In this section, we look at the remainder $\mathbf{w}^{\varepsilon}$. We will proceed in four steps. First in section 2.3.1 we will reduce the initial problem (1.1)-(1.3)-(1.4) for the unknown u^{ε} to a problem for $\mathbf{w}^{\varepsilon}$. Indeed in order to get a homogeneous boundary problem, we will add a corrector to $\mathbf{w}^{\varepsilon}$ and rather work with the resulting term w^{ε}. Some Borel classical arguments will insure the existence of convenient initial data for the resulting reduced problem which means that compatibility conditions either on Γ and on Σ are satisfied. We will prove that the solutions of this nonlinear problems exist not only for a common non trivial time, in fact even till the lifetime T of the profiles $\mathcal{U}$. Moreover these solutions satisfy some estimates uniform with respect to ε. The method lies on a simple Picard iterative scheme (cf. section 2.3.2) and on linear estimates (cf. section 2.3.3). More precisely we will use L^2-type conormal estimates of only the two first normal derivatives, and some Lipschitz estimates. A few carefulness reveals that the presence of the operator $\mathcal{H}$ does not cause any loss of factor ε or any loss of derivatives.

2.3.1 A reduced problem

Since we look for solutions u^ε of (1.1)-(1.3)-(1.4) of the form (2.1) where the functions

$$a^\varepsilon(t,x) \quad := \quad \mathcal{U}(t,x,\frac{\Psi(x)}{\varepsilon}) + \varepsilon\Big(\mathfrak{U}(t,x,\frac{\Phi(x)}{\varepsilon}) + \mathbf{w}^\varepsilon(t,x)\Big)$$

have been constructed above, we look for a problem in term of the remainder $\mathbf{w}^\varepsilon$. In fact, in order to get a homogeneous boundary problem, we choose a function $\rho(t,x) \in H^\infty$ such that

$$\partial_{\mathbf{n}}\rho\,|_\Gamma \quad = \quad -\partial_{\mathbf{n}}\mathfrak{U}(t,x,0)|_\Gamma. \tag{2.28}$$

and will look for remainders $\mathbf{w}^\varepsilon$ of the form $\mathbf{w}^\varepsilon = \rho + w^\varepsilon$. Let us explain why. On the boundary Γ, the function a^ε satisfies:

$$\partial_{\mathbf{n}} a^\varepsilon|_\Gamma = \varepsilon\,\partial_{\mathbf{n}}\mathfrak{U}(t,x,0)|_\Gamma, \tag{2.29}$$

Hence in general a^ε does not satisfy the homogeneous Neumann boundary condition on Γ. We define the function $\tilde{a}^\varepsilon := a^\varepsilon + \varepsilon\rho$. Thus we look for solutions u^ε of (1.1)-(1.3)-(1.4) of the form $u^\varepsilon = a^\varepsilon + \varepsilon\,\mathbf{w}^\varepsilon = \tilde{a}^\varepsilon + \varepsilon\, w^\varepsilon$. Combine (1.3), (2.28) and (2.29) to find a homogeneous Neumann boundary condition on Γ for w^ε:

$$\partial_{\mathbf{n}} w^\varepsilon \quad = \quad 0 \quad \text{on }]0,T[\times\Gamma. \tag{2.30}$$

We now look for an equation on the unknown w^ε. The function $\tilde{a}^\varepsilon$ belongs to $\mathcal{C}^1((0,T)\times\Omega)$ and to $H^\infty_\Sigma(\Omega)$. Moreover, $\tilde{a}^\varepsilon$ satisfies the equation

$$\mathcal{L}(\tilde{a}^\varepsilon,\partial)\,\tilde{a}^\varepsilon = \mathbf{F}\big(\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon)\big) + \varepsilon r^\varepsilon \tag{2.31}$$

where the family $(r^\varepsilon)_\varepsilon$ lies in the set E (defined above Theorem 2.1). The system for the unknown $w^\varepsilon(t,x)$ writes

$$\mathcal{L}(\tilde{a}^\varepsilon + \varepsilon w^\varepsilon,\partial)w^\varepsilon = K(\varepsilon,\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon),w^\varepsilon,\varepsilon\partial_x w^\varepsilon,\mathcal{H}(w^\varepsilon)) + r^\varepsilon \quad \text{in }]0,T[\times\Omega \tag{2.32}$$

where K is a smooth function of its arguments. Let us use more concise notations, and note

$$A^\varepsilon := \big(\,\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon)\,\big) \quad \text{and} \quad W^\varepsilon := \big(\,w^\varepsilon,\varepsilon\partial_x w^\varepsilon,\mathcal{H}(w^\varepsilon)\,\big). \tag{2.33}$$

Then, the Taylor formula shows that the function K has the following form:

$$K(\varepsilon,A^\varepsilon,W^\varepsilon) = G(\varepsilon,A^\varepsilon,\varepsilon W^\varepsilon)W^\varepsilon$$

where G depends smoothly on its arguments (including ε), which will be useful in the sequel.

Following [13] there exist a family $(w^\varepsilon_{\text{init}})_\varepsilon$ of compatible initial conditions for the problem (2.32)-(2.30) which verifies suitable uniform estimates with respect to ε. We choose such a family.

2.3.2 The iterative scheme

We want to solve the problem (2.32),(2.30). We use a simple Picard(-Banach-Caccioppoli) iterative scheme defining a sequence $w^{\varepsilon,\nu}$ which will converge to the solution of the problem. For clarity, we adopt the following more concise notations

$$A^\varepsilon := \big(\,\tilde{a}^\varepsilon,\varepsilon\partial_x\tilde{a}^\varepsilon,\mathcal{H}(\tilde{a}^\varepsilon)\,\big) \quad \text{and} \quad W^{\varepsilon,\nu} := \big(\,w^{\varepsilon,\nu},\varepsilon\partial_x w^{\varepsilon,\nu},\mathcal{H}(w^{\varepsilon,\nu})\,\big).$$

With these notations, the iterative scheme writes

$$\mathcal{L}(\tilde{a}^{\varepsilon} + \varepsilon w^{\varepsilon,\nu}, \partial) w^{\varepsilon,\nu+1} = f^{\varepsilon,\nu} \quad \text{in }]0,T[\times\Omega \tag{2.34}$$

where

$$f^{\varepsilon,\nu} := G(\varepsilon, A^{\varepsilon}, \varepsilon W^{\varepsilon,\nu}) W^{\varepsilon,\nu} + r^{\varepsilon} \tag{2.35}$$

This equation is coupled with the initial and boundary conditions:

$$\partial_{\mathbf{n}} w^{\varepsilon,\nu+1} = 0 \text{ on }]0,T[\times\Gamma \tag{2.36}$$

$$w^{\varepsilon,\nu+1}|_{t=0} = w^{\varepsilon}_{\text{init}}. \tag{2.37}$$

The iterative scheme is initialized with $w^{\varepsilon,0}(t,x) := w^{\varepsilon}_{\text{init}}(x)$.

2.3.3 Estimates for a linear parabolic system

Consider the linear problem

$$\mathcal{L}(\tilde{a}^{\varepsilon} + \varepsilon \mathbf{b}, \partial)\mathbf{u} = f \text{ on }]0,T[\times\Omega \tag{2.38}$$

$$\partial_{\mathbf{n}} \mathbf{u} = 0 \text{ on }]0,T[\times\Gamma, \tag{2.39}$$

We endow the space $H^m_{co}(]0,T[\times\Omega)$ with the usual weighted norm with $\lambda \geq 1$:

$$\|\mathbf{u}\|_{m,\lambda} := \sum_{|\alpha|\leq m\,,\,\alpha\in\mathbb{N}^{1+\mu}} \lambda^{m-|\alpha|} \|e^{-\lambda t} \mathcal{Z}^{\alpha}\mathbf{u}\|_{L^2(]0,T[\times\Omega)}.$$

In order to estimate the initial data, we introduce the similar norms built with the set $\mathcal{T}_0$ instead of $\mathcal{T}$, integrating on Ω instead of $[0,T]\times\Omega$:

$$|\mathbf{u}|_{m,\lambda} := \sum_{|\alpha|\leq m\,,\,\alpha_0=0\,,\,\alpha\in\mathbb{N}^{1+\mu}} \lambda^{m-|\alpha|} \|\mathcal{Z}^{\alpha}\mathbf{u}\|_{L^2(\Omega)}.$$

We will use the following classical Gagliardo-Moser-Nirenberg estimates for conormal derivatives (see [8]).

Lemma 2.2. *Let $m \in \mathbb{N}$. There is $c_m > 0$ such that, for any $a_1, \dots, a_k \in H^m_{co}(]0,T[\times\Omega) \cap L^{\infty}(]0,T[\times\Omega)$, for all multi-index $\alpha_1 \in \mathbb{N}^{\mu+1}, \dots, \alpha_k \in \mathbb{N}^{\mu+1}$, with $|\alpha_1| + \dots + |\alpha_k| \leq m$, for all $\lambda \geq 1$:*

$$\|\mathcal{Z}^{\alpha_1} a_1 \dots \mathcal{Z}^{\alpha_k} a_k\|_{0,\lambda} \leq c_m \sum_{1\leq j\leq k} \Big(\|a_j\|_{m,\lambda} \prod_{i\neq j} \|a_i\|_{\infty} \Big). \tag{2.40}$$

The following proposition gives some ε-conormal estimates for the two first normal derivatives of the solutions of the problem (2.38)-(2.39).

Proposition 2.2. *Let $R > 0$ be an arbitrary constant and $m \geqslant 3$. There exist $C_m(R) > 0$ and $\lambda_m > 0$ such that for σ fixed constant large enough, depending only on the choices of the vector fields $\mathcal{Z}_j$, the following holds true. Assume that*

$$\varepsilon\,(\ \|\mathbf{b}\|_{\infty} + \sum_{0\leq j\leq\mu} \|\mathcal{Z}_j \mathbf{b}\|_{\infty} + \|\varepsilon\partial_x \mathbf{b}\|_{\infty}\ ) \leq R, \tag{2.41}$$

then, for all $\lambda \geq \lambda_m$, the following estimates hold:

$$\begin{aligned} \|\varepsilon\partial_x \mathbf{u}\|_{m,\lambda} + \lambda\|\mathbf{u}\|_{m,\lambda} \leq\ & C_m(R)\ \big[\ \lambda^{-1}\ \|f\|_{m,\lambda} + \ I_{m,\lambda}(\mathbf{u}) \\ & +\ \varepsilon\,(\,\|\varepsilon\partial_x \mathbf{b}\|_{m,\lambda} + \|\mathbf{b}\|_{m,\lambda}\,)\ (\,\|\mathbf{u}\|_{\infty} + \|\varepsilon\partial_x \mathbf{u}\|_{\infty}\,)\ \big], \end{aligned} \tag{2.42}$$

where

$$I_{m,\lambda}(\mathbf{u}) := \sum_{0\le k\le m} |(\partial_t^k \mathbf{u})_{|t=0}|_{m-k,\lambda}.$$

and

$$\begin{aligned}\|(\varepsilon\partial_{\mathbf{n}})^2\mathbf{u}\|_{m,\lambda} \ \le\ C_m(R)\big[\ &\|f\|_{m,\lambda} + \|\mathbf{u}\|_{m+1,\lambda} + \varepsilon\|\mathbf{b}\|_{m+1,\lambda}\,(\|\mathbf{u}\|_\infty + \|f\|_\infty) \\ &+\varepsilon\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m+1,\lambda} + \varepsilon^2\|\mathbf{u}\|_{m+2,\lambda}\ \big].\end{aligned} \tag{2.43}$$

Proof. Step 1. Let us note $\mathbf{v} := e^{-\lambda t}\,\mathbf{u}$, which satisfies

$$\mathcal{L}(\tilde{a}^\varepsilon_{app} + \varepsilon\mathbf{b}, \partial)\mathbf{v} + \lambda\mathbf{v} \ = \ e^{-\lambda t} f \text{ on }]0,T[\times\Omega \tag{2.44}$$

$$\partial_{\mathbf{n}}\mathbf{v} \ = \ 0 \text{ on }]0,T[\times\Gamma. \tag{2.45}$$

$$\mathbf{v} \ = \ w^\varepsilon_{\text{init}} \text{ on } t=0. \tag{2.46}$$

Let us note $\|.\|_{L^2}$ the L^2 norm in $[0,T]\times\Omega$, and $|.|_{L^2}$ the L^2 norm in Ω. Multiplying (2.44) by $\mathbf{v}$ and integrating on $]0,T[\times\Omega$ gives the following estimate, integrating by parts the $\varepsilon^2\Delta_x$ with Green's formula in Ω:

$$\varepsilon^2\|\nabla_x\mathbf{v}\|^2_{L^2} + \lambda\|\mathbf{v}\|^2_{L^2} \le 2\ |((e^{-\lambda t}f, \mathbf{v}))_{L^2}| + |\mathbf{v}(0)|_{L^2}, \tag{2.47}$$

for all $\lambda \ge \lambda_0$ if λ_0 is fixed large enough, and for all $\varepsilon > 0$. In terms of $\mathbf{u}$ it writes

$$\varepsilon^2\|\nabla_x\mathbf{u}\|^2_{0,\lambda} + \lambda\|\mathbf{u}\|^2_{0,\lambda} \le 2\ |((f, \mathbf{u}))_{L^2_\lambda}| + |\mathbf{u}(0)|_{L^2}, \tag{2.48}$$

where L^2_λ is the Hilbert space $L^2(]0,T[\times\Omega, d\mu)$ with the measure $d\mu := e^{-2\lambda t}dtdx$.

Using now the Cauchy-Schwarz inequality in the right hand side, and absorbing in the left hand side the term in $\|v\|^2_{L^2}$ yields the desired estimate for $m=0$ and some constant $c_0 > 0$.

Step 2. We show the inequality by induction on m. Assume it for $m-1$. We apply a tangential operator $\mathcal{Z}^\alpha$ with fields $\mathcal{Z}_i \in \mathcal{T}$ to the system, and $|\alpha| = m$. The function $\mathcal{Z}^\alpha\mathbf{u}$ satisfies the same boundary conditions. The L^2 estimate (2.48) gives, for $\lambda \ge \lambda_0$:

$$\varepsilon^2\|\nabla_x\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} + \lambda\|\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} \le 2|((e^{-\lambda t}\mathcal{Z}^\alpha f + [(\tilde{a}^\varepsilon_{app} + \varepsilon\mathbf{b})\varepsilon^2\Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u}, \mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}|. \tag{2.49}$$

where $[.,.]$ denotes the commutator. Using Cauchy-Schwarz inequality and $2ab \le 2\lambda^{-1}\,a^2 + \lambda b^2/2$ yields:

$$\begin{aligned}\varepsilon^2\|\nabla_x\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} + \frac{\lambda}{2}\|\mathcal{Z}^\alpha\mathbf{u}\|^2_{L^2} \ \le\ &\frac{2}{\lambda}\ \|e^{-\lambda t}\mathcal{Z}^\alpha f\|^2_{L^2} \\ &+ 2\ |((([(\tilde{a}^\varepsilon_{app} + \varepsilon\mathbf{b})\varepsilon^2\Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u}, \mathcal{Z}^\alpha\mathbf{u}))_{L^2_\lambda}|.\end{aligned} \tag{2.50}$$

We need to control the second term in the right hand side of (2.50). The commutator $[\ \tilde{a}^\varepsilon_{app}\,\varepsilon^2\Delta_x, \mathcal{Z}^\alpha]$ writes as a finite sum

$$\varepsilon^2 \sum_{|\beta|\le m+1} a^\varepsilon_\beta(t,x)\mathcal{Z}^\beta + \varepsilon\sum_{|\gamma|\le m} b^\varepsilon_\gamma(t,x)\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^\gamma + \sum_{|\delta|\le m-1} c^\varepsilon_\delta(t,x)(\varepsilon\partial_{\mathbf{n}})^2\mathcal{Z}^\delta \tag{2.51}$$

where the coefficients a^ε_β , b^ε_γ, c^ε_δ are bounded functions satisfying

$$\sup_{\varepsilon\in]0,1]} \|\varepsilon\partial_{\mathbf{n}}a^\varepsilon_\beta\|_{L^\infty(\Omega)} + \|\varepsilon\partial_{\mathbf{n}}b^\varepsilon_\gamma\|_{L^\infty(\Omega)} + \|\varepsilon\partial_{\mathbf{n}}c^\varepsilon_\delta\|_{L^\infty(\Omega)} < \infty \tag{2.52}$$

for all β, γ, δ, because (2.52) holds clearly if we replace $L^\infty(\Omega)$ by $L^\infty(\Omega_+)$ or by $L^\infty(\Omega_-)$, and because $\tilde{a}^\varepsilon_{app}$ is in $H^1(\Omega)$ for all $\varepsilon > 0$. Hence we are led to control the corresponding three sort of terms:

$$\varepsilon^2((\, a^\varepsilon_\beta \mathcal{Z}^\beta \mathbf{u} \, , \, \mathcal{Z}^\alpha \mathbf{u} \,))_{L^2_\lambda}, \;\; \varepsilon((\, b^\varepsilon_\gamma (\varepsilon \partial_\mathbf{n}) \mathcal{Z}^\gamma \mathbf{u} \, , \, \mathcal{Z}^\alpha \mathbf{u} \,))_{L^2_\lambda}, \;\; ((\, c^\varepsilon_\delta (\varepsilon \partial_\mathbf{n})^2 \mathcal{Z}^\delta \mathbf{u} \, , \, \mathcal{Z}^\alpha \mathbf{u} \,))_{L^2_\lambda}, \tag{2.53}$$

where $|\beta| \leq m+1$, $|\gamma| \leq m$, $|\delta| \leq m-1$. The first two terms in (2.53) are simply controlled by $\delta \|\varepsilon \nabla_x \mathbf{u}\|^2_{m,\lambda} + C_\delta \, \delta^{-1} \, \|\mathbf{u}\|^2_{m,\lambda}$ for δ arbitrarily small, and C_δ being a constant depending on δ, but independent of ε. For the third term one uses an integration by parts (by Green's formula) of the field $\partial_\mathbf{n}$ to show that this term writes as a sum of terms of the form

$$d^\varepsilon \; \varepsilon^{2-j-j'} (((\varepsilon \partial_\mathbf{n})^j \mathcal{Z}^\delta \mathbf{u}, (\varepsilon \partial_\mathbf{n})^{j'} \mathcal{Z}^\alpha \mathbf{u}))_{L^2_\lambda}$$

where $|\delta| \leq m-1$, $j, j' \in \{0, 1\}$, and d^ε is a bounded function (uniformly in ε) since all the boundary terms terms vanishes: $\partial_\mathbf{n} \mathcal{Z}^\alpha \mathbf{u}|_{\partial\Omega} = 0$, for all $\alpha \in \mathbb{R}^\mu$. It follows that the third term in (2.53) is controlled by $C\lambda^{-1} \|\varepsilon \nabla_x \mathbf{u}\|^2_{m,\lambda} + C\|\mathbf{u}\|^2_{m,\lambda}$ for a constant C independent of ε, and all $\lambda \geq 1$. Hence, by choosing a $\delta > 0$ arbitrarily small, and $\lambda_1 > 0$ large enough, there holds

$$| \; ((\; [\tilde{a}^\varepsilon_{app} \varepsilon^2 \Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u} \, , \, \mathcal{Z}^\alpha \mathbf{u} \;))_{L^2_\lambda} \; | \; \leq \; \delta \|\varepsilon \nabla_x \mathbf{u}\|^2_{m,\lambda} + c_m \|\mathbf{u}\|^2_{m,\lambda}$$

for all $\lambda \geq \lambda_1$, and for all $\varepsilon \in]0, 1]$, with a constant c_m independent of ε.

We need now to estimate the term

$$(([\; \varepsilon \mathbf{b} \, \varepsilon^2 \Delta_x, \mathcal{Z}^\alpha] \wedge \mathbf{u}, \mathcal{Z}^\alpha \mathbf{u}))_{L^2_\lambda}. \tag{2.54}$$

The commutator $[\; \mathbf{b} \, \varepsilon^2 \Delta_x, \mathcal{Z}^\alpha]$ writes as a finite sum

$$\begin{aligned} &\varepsilon^2 \sum_{|\beta| \leq m, |\beta'| \leq m+1, |\beta| + |\beta'| \leq m+2} a_{\beta,\beta'} (\mathcal{Z}^\beta \mathbf{b}) \mathcal{Z}^{\beta'} \\ &+ \varepsilon \sum_{|\gamma| \leq m, |\gamma'| \leq m, |\gamma| + |\gamma'| \leq m+1} b_{\gamma,\gamma'} (\mathcal{Z}^\gamma \mathbf{b}) (\varepsilon \partial_\mathbf{n}) \mathcal{Z}^{\gamma'} \\ &+ \sum_{|\delta| \leq m, |\delta'| \leq m-1, |\delta| + |\delta'| \leq m} c_{\delta,\delta'} (\mathcal{Z}^\delta \mathbf{b}) (\varepsilon \partial_\mathbf{n})^2 \mathcal{Z}^{\delta'} \end{aligned}$$

where $a_{\beta,\beta'}, b_{\gamma,\gamma'}, c_{\delta,\delta'}$ are smooth fonctions on $\overline{\Omega}$. Hence to control the term (2.54) we are led to estimate tri-linear terms in $(\mathbf{b}, \mathbf{u}, \mathbf{u})$ of the following form (where $d\mu := e^{-2\lambda t} dt dx$):

$$\varepsilon^2 \int_{]0,T[\times\Omega} a_{\beta,\beta'} \mathcal{Z}^\beta \mathbf{b} . \mathcal{Z}^{\beta'} \mathbf{u}_i . \mathcal{Z}^\alpha \mathbf{u}_j \; d\mu, \quad |\beta| \leq m, |\beta'| \leq m+1, |\beta| + |\beta'| \leq m+2 \tag{2.55}$$

$$\varepsilon \int_{]0,T[\times\Omega} b_{\gamma,\gamma'} \; \mathcal{Z}^\gamma \mathbf{b} \; . \; \varepsilon \partial_\mathbf{n} \mathcal{Z}^{\gamma'} \mathbf{u}_i \; . \; \mathcal{Z}^\alpha \mathbf{u}_j \; d\mu, \quad |\gamma| \leq m, |\gamma'| \leq m, |\gamma| + |\gamma'| \leq m+1 \tag{2.56}$$

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'} \; \mathcal{Z}^\delta \mathbf{b} . \, (\varepsilon \partial_\mathbf{n})^2 \mathcal{Z}^{\delta'} \mathbf{u}_i \, . \mathcal{Z}^\alpha \mathbf{u}_j \; d\mu, \quad |\delta| \leq m, |\delta'| \leq m-1, |\delta| + |\delta'| \leq m, \tag{2.57}$$

where the $\mathbf{u}_i$ are the components of the vector $\mathbf{u}$. Let us treat the term (2.57). By the green formula, the integral can be written as a sum of integrals of the form

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'} \; \mathcal{Z}^\delta \varepsilon \partial_\mathbf{n} \mathbf{b} . \mathcal{Z}^{\delta'} \varepsilon \partial_\mathbf{n} \mathbf{u}_i . \mathcal{Z}^\alpha \mathbf{u}_j \; d\mu \tag{2.58}$$

$$\int_{]0,T[\times\Omega} c_{\delta,\delta'} \; \mathcal{Z}^\delta \mathbf{b} . \, \mathcal{Z}^{\delta'} \varepsilon \partial_\mathbf{n} \mathbf{u}_i . \, \mathcal{Z}^\alpha \varepsilon \partial_\mathbf{n} \mathbf{u}_j \; d\mu, \tag{2.59}$$

$$\varepsilon \int_{]0,T[\times\Omega} d_{\delta,\delta'} \; \mathcal{Z}^\delta \mathbf{b} . \, \mathcal{Z}^{\delta'} \varepsilon \partial_\mathbf{n} \mathbf{u}_i \, . \mathcal{Z}^\alpha \mathbf{u}_j \; d\mu, \tag{2.60}$$

and other terms involving lower order derivatives easy to control. The term (2.58) is controlled by

$$c\|\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^{\delta}\mathbf{b}\varepsilon\partial_{\mathbf{n}}\mathcal{Z}^{\delta'}\mathbf{u}_i\|_{0,\lambda}\ \|\mathbf{u}_j\|_{m,\lambda},$$

which is bounded by using the Gagliargo-Nirenberg-Moser estimate by

$$c\big(\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_{\infty}+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{b}|_{\infty}\ \big)\ \|\mathbf{u}\|_{m,\lambda}$$

and hence by

$$c(1+R)\big(\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_{\infty}+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}\big)\ \|\mathbf{u}\|_{m,\lambda}.$$

For the term (2.59) there are two cases. The first case is when $\delta=0$. In that case the integral is bounded by

$$c\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{u}_i\|_{m-1,\lambda}\ \|\mathbf{u}_j\|_{m,\lambda}\ \le\ \lambda^{-1}\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|^2_{m,\lambda}.$$

The second case is when $|\delta|\geq 1$. In that case we write $\mathcal{Z}^{\delta}\mathbf{b}=\mathcal{Z}^{\delta''}\mathcal{Z}_k\mathbf{b}$ and apply the Gagliardo-Nirenberg-Moser inequality with $\mathcal{Z}\mathbf{b}$ in L^{∞}. The term in bounded by

$$c\big(\ \|\mathcal{Z}\mathbf{b}\|_{m-1,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_{\infty}+\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m-1,\lambda}|\mathcal{Z}\mathbf{b}|_{\infty}\ \big)\ \|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}$$

and hence by

$$c\ \|\mathbf{b}\|_{m,\lambda}|\varepsilon\partial_{\mathbf{n}}\mathbf{u}|_{\infty}\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|_{m,\lambda}+cR\lambda^{-1}\|\varepsilon\partial_{\mathbf{n}}\mathbf{u}\|^2_{m,\lambda}$$

The next terms like (2.60) are easier to treat in the same way, and are bounded by the same terms. The term (2.57) was the more delicate to estimate. The terms (2.56) and (2.55) are simpler and can be treated in a similar way. Replacing in the right hand side of (2.49) and summing over all the possible operators $\mathcal{Z}^{\alpha}$ gives the desired estimate, and the proposition is proved. □

2.3.4 Iteration

Now classical arguments show the convergence of the iterative scheme if $\varepsilon\in]0,\varepsilon_0]$ and ε_0 is small enough. We describe the main lines (see [13]). Let us fix an integer $m>4$, and note

$$R:=1+\sup_{0<\varepsilon<1}\{\varepsilon\,(\ \|w^{\varepsilon,0}\|_{\infty}+\sum_{0\leq j\leq\mu}\|\mathcal{Z}_j w^{\varepsilon,0}\|_{\infty}+\|\varepsilon\partial_x w^{\varepsilon,0}\|_{\infty}\ )\}.$$

Proposition 2.3. *Let be given $\lambda>1$. Then there exists $h>1$ such that for $\varepsilon_0>0$ small enough, for all $\nu\in\mathbb{N}$, for all $\varepsilon\in]0,\varepsilon_0]$, there hold*

$$\|w^{\varepsilon,\nu}\|_{\infty}+\sum_{0\leq j\leq\mu}\|\mathcal{Z}_j w^{\varepsilon,\nu}\|_{\infty}+\|\varepsilon\partial_x w^{\varepsilon,\nu}\|_{\infty}<R\varepsilon^{-1}\tag{2.61}$$

and

$$\|w^{\varepsilon,\nu}\|_{m,\lambda}+\|\varepsilon\partial_{\mathbf{n}}w^{\varepsilon,\nu}\|_{m,\lambda}<h.\tag{2.62}$$

Proof. For h large enough, the inequalities (2.61) and (2.62) are satisfied for $\nu=0$. Now suppose that $w^{\varepsilon,\nu}$ satisfies (2.61), (2.62). We want to prove that $w^{\varepsilon,\nu+1}$ also satisfies (2.61), (2.62). The proposition 2.2 gives a constant $C_m(R)$ and the inequality (2.42) holds with $\mathbf{u}=w^{\varepsilon,\nu+1}$, $\mathbf{b}=w^{\varepsilon,\nu}$, and $f=f^{\varepsilon,\nu}$ defined in (2.35). In order to control the right hand side of (2.34), we need a control of $\|\mathcal{H}(w^{\varepsilon,\nu})\|_{\infty}$ and of $\|\mathcal{H}(w^{\varepsilon,\nu})\|_{m,\lambda}$, which is a consequence of the following lemma.

Lemma 2.3. *Let $m \in \mathbb{N}$. There exists $c > 0$ such that for all $\lambda \geq 1$,*

$$\|\mathcal{H}(v)\|_{m,\lambda} + \|\varepsilon\partial_{\mathbf{n}}\mathcal{H}(v)\|_{m-1,\lambda} \leq c(\|v\|_{m,\lambda} + \|\varepsilon\partial_{\mathbf{n}}v\|_{m-1,\lambda}). \tag{2.63}$$

Proof. We note $E(\partial) := (\mathrm{div}\ , \mathrm{curl}\ )$ the operator from $[\mathcal{S}'(\mathbb{R}^3)]^3$ to $[\mathcal{S}'(\mathbb{R}^3)]^4$. We denote by $E^{-1}(\partial)$ the inverse operator. Then $u = E^{-1}(\partial)f$, is defined by $\hat{u}(\xi) = -i|\xi|^{-2}\big(\hat{a}(\xi)\xi - \xi \wedge \hat{b}(\xi)\big)$ where $\hat{f}(\xi) = (\hat{a}(\xi), \hat{b}(\xi)) \in \mathbb{R} \times \mathbb{R}^3$. Thus $\hat{u}(\xi) = M(\xi)\hat{f}(\xi)$, where $M(\xi)$ is a 3×4 matrix whose entries are *rational functions of ξ homogeneous of degree* -1. Let us fix $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\chi(\xi) = 0$ when $|\xi| \leq 1$ and $\chi(\xi) = 1$ when $|\xi| \geq 2$, and call $P(D)$ and $R(D)$ the operators from $[\mathcal{S}'(\mathbb{R}^3)]^4$ to $[\mathcal{S}'(\mathbb{R}^3)]^3$ defined by $P(D)f := \mathcal{F}^{-1}\big(\chi M\hat{f}\big)$ and $R(D)f := \mathcal{F}^{-1}\big((1-\chi)\hat{f}\big)$ where $\mathcal{F}^{-1}$ means the inverse Fourier transform. In the sequel we will simply note $\mathcal{S}'(\mathbb{R}^3)$ and $L^2(\tilde{\Omega})$ instead of $\big[\mathcal{S}'(\mathbb{R}^3)\big]^4$ and $\big[L^2(\tilde{\Omega})\big]^4$, meaning that we talk about the *components* of the vector valued functions, the (finite) number of components being understood. We have $E^{-1}(\partial) = P(D) + R(D)$. The operator $P(D)$ is a special case of classical pseudo-differential operator of class $S^1_{-1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$, elliptic, and $R(D)$ is an infinitely smoothing operator of class $S^{-\infty}_{1,0}(\mathbb{R}^3 \times \mathbb{R}^3)$.

Let us now take into account the t coordinate. Let us note $\tilde{\Omega} =]0,T[\times\Omega$, $\tilde{\Gamma} =]0,T[\times\Gamma$ and $\tilde{\Sigma} =]0,T[\times\Sigma$. We extend the actions of P and R to the spaces of functions or distributions which depend also on t like $L^2(\tilde{\Omega})$ or $C\big([0,T], \mathcal{S}'(\mathbb{R}^3)\big)$, by considering t as a parameter so that $Pu(t,x) := P(D)u(t,.)(x)$. Let $v \in H^m_{co}(\tilde{\Omega}; \mathbb{R}^4)$ such that $\partial_{\mathbf{n}}v \in H^{m-1}_{co}(\tilde{\Omega}; \mathbb{R}^4)$. Then $\mathcal{H}(v) = u_{|\tilde{\Omega}}$ where $u \in L^2([0,T] \times \mathbb{R}^3)$ is defined by $E(\partial)u = \overline{E(\partial)v} + (v_{|\tilde{\Gamma}}.\mathbf{n}) \otimes \delta_{\tilde{\Gamma}}$, where the notation $\overline{V}$ means the extension of V by 0 to $[0,T] \times \mathbb{R}^3$ (or to $\mathbb{R}^3$, depending on the context).

Let us note $f := \overline{E(\partial)v}$, which is in $H^{m-1}_{co}(]0,T[\times\mathbb{R}^3)$, and $g = (v.\mathbf{n})_{|\Gamma}$. This trace is well defined since by assumption $v \in H^1(\tilde{\Omega})$, and using local coordinates patches one sees that $g \in H^{m-\frac{1}{2}}(\tilde{\Gamma})$, the usual Sobolev spaces. The operator $P(D)$ satisfies the *transmission property* (introduced by Boutet de Monvel [1], [2]) on Ω and on $\mathbb{R}^3 \setminus \Omega$ because its symbol is a rational function of ξ, which is a sufficient condition to satisfy the transmission condition. The transmission property has been also studied and used by Grubb, and we also refer to papers [6] and [7]. To avoid many repetitions, we will note in what follows $\Omega_1 := \Omega$ and $\Omega_2 = \mathbb{R}^3 \setminus \Omega$. Since $P(D)$ is elliptic of order 1, the transmission property implies (see [6] and [7]) that if $v \in H^s(\Omega)$ then for $j = 1,2$, $\big(P(D)\overline{v}\big)_{|\Omega_j} \in H^{s+1}(\Omega_j)$. Let us note note $u_{(j)} = u_{|\tilde{\Omega}_j}$, for $j = 1,2$, so that $\mathcal{H}(v) = u_{(1)} \in L^2(\tilde{\Omega})$. Using the notations of [1], [6], [7],

$$u_{(j)} = \big(E^{-1}(D)\overline{v}\big)_{|\tilde{\Omega}_{(j)}} = P(D)^{(j)}f + K^{(j)}_\Gamma(g) + R(D)^{(j)}E(\partial)\overline{v}, \tag{2.64}$$

where $P(D)^{(j)}f = (P(D)f)_{|\tilde{\Omega}_{(j)}}$, $R(D)^{(j)}\overline{v} = (R(D)\overline{v})_{|\tilde{\Omega}_{(j)}}$ and where $K^{(j)}_\Gamma(g) = \big(P(D)\big(g \otimes \delta_\Gamma\big)\big)_{|\tilde{\Omega}_{(j)}}$ is the "Poisson operator":

$$K^{(j)}_\Gamma : H^s(\Gamma) \to H^{s+1/2}(\Omega^{(j)}), \tag{2.65}$$

(linear continuous), extended to functions depending on t as a parameter. (See theorems 2.4 and 2.5 of [7]).

Let us now prove the lemma. First of all, $\partial_t^m\mathcal{H}(v) = \mathcal{H}(\partial_t^m v)$ is in $L^2(\tilde{\Omega})$ because $\partial_t^m v \in L^2(\tilde{\Omega})$ and $\mathcal{H}$ acts on $L^2(\tilde{\Omega})$. It is also easy to show that $\partial_t^{m-1}\mathcal{H}(v) \in H^1(\tilde{\Omega})$: by assumption, for any $t \in [0,T]$, $\partial_t^{m-1}v(t,.) \in H^1(\Omega)$, hence $\mathcal{H}(\partial_t^{m-1}\overline{v})(t,.) \in H^1(\Omega)$ because $\partial_t^{m-1}\overline{v}(t,.)$ is piecewise-H^1 and because of the properties of $\mathcal{H}$. Hence $\partial_x\partial_t^{m-1}\mathcal{H}(v) \in L^2(\tilde{\Omega})$ and since we already know that $\partial_t^m\mathcal{H}(v) \in L^2(\tilde{\Omega})$ we have proved that $\partial_t^{m-1}\mathcal{H}(v) \in H^1(\tilde{\Omega})$.

Let us show now that $\mathcal{Z}_j \partial_t^{m-2} \mathcal{H}(v) \in H^1(\tilde\Omega)$ for $j = 1, \ldots, \mu$. We have $E(\partial)u = f + g \otimes \delta_\Gamma$. Since $E(\partial)$ is elliptic (as an operator in $\mathcal{S}'(\mathbb{R}^3)$, but not in $\mathcal{S}'(\mathbb{R}^4)$), we can express the normal derivatives of u in term of tangential derivatives and of $E(\partial)u$, and this implies that the commutator $[E(\partial), \mathcal{Z}_j]u$ writes

$$[E(\partial), \mathcal{Z}_j]u = \sum_1^\mu A_j \mathcal{Z}_j u + A_0 f + B g \otimes \delta_\Gamma \tag{2.66}$$

where A_j, B are matrices with $\mathcal{C}_b^\infty$ entries (depending on the fields $\mathcal{Z}_j$). It follows that

$$E(\partial)\mathcal{Z}_j u = \sum_{|\alpha|\le 1} M_\alpha \mathcal{Z}^\alpha f + \sum_{|\alpha|\le 1} N_\alpha (\mathcal{Z}^\alpha g) \otimes \delta_\Gamma$$

with $\mathcal{C}_b^\infty(\mathbb{R}^3)$ matrices M_α, N_α, and applying ∂_t^{m-2} gives:

$$E(\partial)\mathcal{Z}_j \partial_t^{m-2} u = \sum_{|\alpha|\le 1} M_\alpha . \mathcal{Z}^\alpha \partial_t^{m-2} f + \sum_{|\alpha|\le 1} N_\alpha . (\mathcal{Z}^\alpha \partial_t^{m-2} g) \otimes \delta_\Gamma \tag{2.67}$$

Now $\mathcal{Z}^\alpha \partial_t^{m-2} f \in L^2(\tilde\Omega)$, because $f = \overline{E(\partial)v}$, and the transmission property implies that for every $t \in [0,T]$, the function $P(D)^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-1} f\big)(t,.)$ is in $H^1\big(\Omega_{(j)}\big)$. This implies that $\partial_x P(D)^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-2} f\big) \in L^2(\tilde\Omega)$ and since we already know that $\partial_t P(D)^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-2} f\big) \in L^2(\tilde\Omega)$ from the previous case, we deduce that for $j = 1, 2$ the functions $P(D)^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-2} f\big)$ is in $H^1\big(\tilde\Omega_{(j)}\big)$. Concerning the boundary term in (2.67), since $g \in H^{m-\frac{1}{2}}(\tilde\Gamma)$ we know that $\mathcal{Z}^\alpha \partial_t^{m-2} g \in H^{1/2}(\Gamma)$ and the property (2.65) implies that, for all $t \in [0,T]$, the functions $K_\Gamma^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-2} g\big)(t,.)$ is in $H^1\big(\Omega_{(j)}\big)$. By the same way as before we deduce that for $j = 1, 2$ the functions $K_\Gamma^{(j)}\big(\mathcal{Z}^\alpha \partial_t^{m-2} g\big)$ is in $H^1\big(\tilde\Omega_{(j)}\big)$. Now, applying $E(\partial)^{-1} = P(D) + R(D)$ to the equation (2.67) gives $\mathcal{Z}_j \partial_t^{m-2} u_{(j)} \in H^1\big(\tilde\Omega_{(j)}\big)$ as claimed. Then, the proof can be continued by induction in the same way. □

The lemma 2.3, together with the Gagliardo-Nirenberg-Moser estimates and the induction assumption, implies that (like the majoration of the term (5.25) in paper [13]):

$$\|f^{\varepsilon,\nu}\|_{m,\lambda} \le c(R)(\|w^{\varepsilon,\nu}\|_{m,\lambda} + \|\varepsilon \partial_x w^{\varepsilon,\nu}\|_{m,\lambda}) < c(R)\rho(\lambda). \tag{2.68}$$

Hence, the proposition 2.2 implies that

$$\begin{aligned} \|\varepsilon \partial_x w^{\varepsilon,\nu+1}\|_{m,\lambda} + \lambda \|w^{\varepsilon,\nu+1}\|_{m,\lambda} \le C_m(R)\big[&\lambda^{-1} c(R)\rho(\lambda) \\ &+ R(\|w^{\varepsilon,\nu+1}\|_\infty + \|\varepsilon \partial_x w^{\varepsilon,\nu+1}\|_\infty) + I_{m,\lambda}(w^{\varepsilon,\lambda})\big]. \end{aligned} \tag{2.69}$$

We now use the following Sobolev inequalities:

$$\begin{aligned} \varepsilon^{1/2}\|\mathbf{u}\|_\infty &\le e^{\sigma\lambda}(\|\mathbf{u}\|_{m,\lambda} + \|\varepsilon \partial_{\mathbf{n}} \mathbf{u}\|_{m,\lambda}), \\ \varepsilon^{1/2}\|\varepsilon \partial_{\mathbf{n}} \mathbf{u}\|_\infty &\le e^{\sigma\lambda}(\|\varepsilon \partial_{\mathbf{n}} \mathbf{u}\|_{m,\lambda} + \|(\varepsilon \partial_{\mathbf{n}})^2 \mathbf{u}\|_{m,\lambda}). \end{aligned}$$

By taking λ large enough et $\varepsilon > 0$ small enough the inequality (2.61) is also satisfied for $w^{\varepsilon,\nu+1}$ and the proof by induction is complete. □

Now by extracting a convergent subsequence it is a classical argument to show the convergence in $L^2(]0,T[\times\Omega)$ of $w^{\varepsilon,\nu}$ to a solution w^ε of the non linear problem which satisfies the same estimates (2.61), (2.62). This concludes the proof of Theorem 2.1. □

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